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SYNTHESIS OF OPTIMAL GUARANTEED CONTROL IN FINITE-DIFFERENCE APPROXIMATION SCHEMES†

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Procedures are proposed for synthesizing of optimal controls using the method of extremal aiming in the direction of the gradients of approximations of the value functions in problems with terminal and integral discount functionals. The value functions are approximated by finite-difference operators for a Hamilton–Jacobi equation, using constructions of sub- and superdifferentials of local convex and concave closures. The dependence of the approximation stepsize on the phase space and the time interval is investigated. It is shown that the trajectories generated by a control synthesized by the proposed procedures are indeed optimal. The possibilities of the computational methods are illustrated by an example: the solution of a bimatrix evolutionary game with non-linear dynamics generalizing the classical replicator models.

It is well known that if a value function is differentiable, it is a classical solution of a Hamilton–Jacobi equation. An optimal control procedure may then be constructed as an extremal in the direction of the gradients of the value function. If the value function is not smooth, optimal strategies may be constructed by the method of extremal displacement to accompanying points of its local extrema [1, 2]. Use may also be made of the principle of extremal aiming in the direction of quasi-gradients, defined using the Yosida–Moreau transformation [3]. These methods require either exact knowledge or a highly accurate approximation of the value function. The construction of a value function is a problem in itself, which can be solved in the context of the theory of generalized solutions of Hamilton–Jacobi equations [4, 5]. One method for constructing the value function in that theory uses finite-difference approximation schemes [6–11]. The finite-difference operators involved use generalized gradients of various types [9–11].

Below we propose an algorithm that combines a finite-difference approximation scheme for constructing the values of a value function with the method of optimal aiming in the direction of the generalized gradients. Interpolations of the extremal values of the control parameters, computed at the mesh points, are considered. At the same time, the question of the relationship between the approximation step sizes of the time interval and the phase space is considered. Modifications of the approximation schemes are proposed for problems with an integral discount functional [11, 12].

1. APPROXIMATION SCHEMES IN A TERMINAL PROBLEM

Consider the Cauchy problem for the Hamilton–Jacobi equation

$$\frac{\partial w}{\partial t}+H\left(t,x,\frac{\partial w}{\partial x}\right)=0,\quad (t,x)=T\times R^n,\quad T=(t_0,\theta) \tag{1.1}$$

$$w(\theta,x)=\sigma(x),\quad x\in R^n \tag{1.2}$$

Let us assume that this boundary-value problem is associated with a guaranteed-control problem for a dynamical system

$$\dot{x}=f(t,x,p,q)=A(t,x)+B(t,x)p+C(t,x)q \tag{1.3}$$

$$t\in T,\quad x\in R^n,\quad p\in P\subset R^p,\quad q\in Q\subset R^q$$

with terminal payoff functional

$$\gamma(x(\cdot))=\sigma(x(\theta)) \tag{1.4}$$

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where x is the n-dimensional system vector, p is the control and q is a perturbation. The sets P and Q are convex and compact.

The function $H(t, x, s)$: $T \times R^n \times R^n \to R$ in equation (1.1) is a Hamiltonian for system (1.3), that is, it is related to the dynamics $f(t, x, p, q)$ by the relation

$$H(t, x, s) = \langle s, A(t, x)\rangle + \min_{p\in P}\langle s, B(t, x)p\rangle + \max_{q\in Q}\langle s, C(t, x)q\rangle \tag{1.5}$$

Assume that the right-hand side $f(t, x, p, q)$ of system (1.3) satisfies a Lipschitz condition with constant L with respect to the variables t and x and conditions for the continuability of solutions.

We define a compact domain $G_r \in T \times R^n$, in which we will consider Eq. (1.1) and system (1.3), by the following invariance condition: if $(t_0, x_0) \in G_r$, then $(t, x_0 + (t - t_0)B_r) \in G_r$ for all $t \in T$, $B_r = \{b \in R^n: \| b \| \leq r\}$.

Here

$$r > K, \quad K = \max_{(t,x,p,q)\in G\times P\times Q} \| f(t, x, p, q)\| \tag{1.6}$$

is the maximum velocity of the system, defined on a closed set G which satisfies a strong invariance condition with respect to the differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad t \in T, \quad x(t_0) = x_0$$

$$F(\tau, y) = \{f \in R^n: \ f = f(\tau, y, p, q), \ p \in P, \ q \in Q\}, \quad (\tau, y) \in T \times R^n$$

By virtue of the above conditions imposed on the right-hand side of system (1.3), the Hamiltonian $(t,x,s) \to H(t,x,s)$: $G_r \times R^n \to R$ satisfies a Lipschitz condition and the condition of positive homogeneity with respect to the variable s.

A fundamental role in solving problem (1.3), (1.4) is assigned to the value function $(t, x) \to w(t, x)$: $G_r \times R^n \to R$, which is defined for an initial position (t_0, x_0), positional strategies $U = U(t, x)$ and the corresponding trajectories $x(\cdot) \in X(t_0, x_0, U)$ by the formula

$$w(t_0, x_0) = \min_U \max_{x(\cdot)\in X(t_0,x_0,U)} \sigma(x(\theta)) \tag{1.7}$$

Note that by the alternative theorem [1, 2] the function (1.7) has a saddle point

$$w(t_0, x_0) = \min_V \max_{y(\cdot)\in Y(t_0,x_0,V)} \sigma(y(\theta))$$

where $y(\cdot) \in Y(t_0, x_0, V)$ are the trajectories generated by a perturbation $V = V(t, x)$ from the initial position (t_0, x_0).

We recall that the value function is a generalized (minimax or viscosity) solution of the Hamilton–Jacobi equation (1.1); for the theory of such solutions, see [4, 5].

2. OPTIMAL CONTROL PROCEDURES

To construct optimal control procedures we will have to approximate the value function. We define a finite-difference operator CU, which will be interpreted as a minimax construction

$$CU(t, \Delta, u)(x) = \min_{y\in O(x,K\Delta)} \min_{s\in D^*G(y)} \{\Delta H(t, x, s) + G(y) - \langle s, y - x\rangle\} \quad (t, x) \in G_r \tag{2.1}$$

on local concave closures $y \to G(y)$: $\overline{O}(x, r, \Delta) \to R$ for a function $y \to u(y)$ approximating the value function $y \to w(t + \Delta, y)$ in the interval $(t, t + \Delta)$. The set $D^*G(y)$ is the superdifferential of the function G at the point y

$$D^*G(y) = \{s \in R^n: G(\bar{y}) - G(y) \leq \langle s, \bar{y} - y\rangle, \quad \bar{y} \in \overline{O}(x, r\Delta)\}, \quad y \in \overline{O}(x, K\Delta)$$

The sets $\overline{O}(x, r\Delta)$, $\overline{O}(x, K\Delta)$ are closed neighbourhoods of the point x of radii $r\Delta$, $K\Delta$, $r > K$.

Suppose

$$\Gamma = \{t_0 < t_1 < \ldots < t_N = \theta\}, \quad \Delta = t_{i+1} - t_i, \quad i = 0, 1, \ldots, N-1$$

is an arbitrary partition of the time interval T. We will now analyse the approximation scheme with the finite-difference operator CU of (2.1) for the partition Γ of T

$$u(\theta, x) = \sigma(x) \tag{2.2}$$

$$u(t_i, x) = CU(t_i, \Delta, u(t_{i+1}, \cdot))(x) \tag{2.3}$$

$$(t_i, x), (t_{i+1}, x) \in G_r, \quad i = 0, \ldots, N-1$$

Let us assume that, in the scheme (2.2), (2.3) with operator CU of (2.1), we have constructed an approximation y for the value function w of (1.7) at all points $(t, x) \in G_r, t \in \Omega$. We will define the value of the positional strategy $U^* = U^*(t, x)$ according to the principle of extremal aiming in the direction of generalized gradients—the subgradients s^* of the local concave closure G of u

$$U^* = U^*(t, x) = \arg\min_{p \in P} \{\langle s^*, B(t, x)p\rangle\} \tag{2.4}$$

$$s^* = s^*(t, x, y^*) = \arg \min_{s \in D^*G(y^*)} \{\Delta H(t, x, s) + G(y^*) - \langle s, y^* - x\rangle\} \tag{2.5}$$

$$y^* = y^*(t, x) = \arg \min_{y \in \bar{O}(x, K\Delta)} \min_{s \in D^*G(y)} \{\Delta H(t, x, s) + G(y) - \langle s, y - x\rangle\} \tag{2.6}$$

We will establish relations for the values

$$u(t, x), \quad u(t + \Delta, y(t, x, \Delta, U^*, q))$$

of the approximation function u on an element $y(\cdot)$ of the Euler polygon

$$y(t, x, \Delta, U^*, q) = x + \Delta(A(t, x) + B(t, x)U^* + C(t, x)q), \quad q \in Q$$

Lemma 2.1. The control U^* of (2.4) satisfies the inequality

$$\max_{q \in Q} u(t + \Delta, y(t, x, \Delta, U^*, q)) \leqslant u(t, x) \tag{2.7}$$

Proof. Consider the function G^* conjugate to the local concave closure $y \to G(y)$: $\bar{O}(x, K\Delta) \to R$. Since G and G^* are concave functions, the following chain of inequalities is true for any $q \in Q$

$$\begin{aligned} &u(t + \Delta, y(t, x, \Delta, U^*, q)) \leqslant G(y(t, x, \Delta, U^*, q)) \leqslant \max_{q \in Q} G(y(t, x, \Delta, U^*, q)) = \\ &= \max_{q \in Q} \inf_{s \in R^n} \{\langle s, y(t, x, \Delta, U^*, q)\rangle - G^*(s)\} = \min_{y \in \bar{O}(x, K\Delta)} \min_{s \in D^*G(y)} \{\langle s, x\rangle + \Delta(\langle s, A(t, x)\rangle + \\ &+ \langle s, B(t, x)U^*\rangle + \max_{q \in Q}\langle s, C(t, x)q\rangle) - \langle s, y\rangle - G(y)\} \end{aligned} \tag{2.8}$$

Using the definition of the positional control U^* (2.4)–(2.6), we obtain inequality (2.7).

Having inequality (2.7) for the function u on an element $y(\cdot)$ of the Euler polygon, we can estimate the quality of the entire trajectory $x(\cdot)$ generated by the strategy U^*.

Fix an initial position (t_0, x_0). Consider the Euler polygon

$$x(\cdot) = \{x(t, t_0, x_0, U^*, q(\cdot)), \quad t \in \Gamma \cap T\} \tag{2.9}$$

generated by the strategy U^* of (2.4) and an arbitrary perturbation $t \to q(t)$

$$x(t_{i+1}) = x(t_i + \Delta) = x(t_i) + \Delta(A(t_i, x(t_i)) + B(t_i, x(t_i))U^* + C(t_i, x(t_i))q(t_i)),$$

$$t_i, t_{i+1} \in \Gamma, \quad x(t_0) = x_0$$

Inequality (2.7) implies the following proposition for the trajectory $x(\cdot)$ of (2.9).

Theorem 2.1. For any partitions Γ, initial positions (t_0, x_0) and arbitrary perturbations $t \to q(t)$, the trajectory $x(\cdot)$ generated by the strategy U^* of (2.4) satisfies the estimate

$$\sigma(x(\theta)) \leq u(t_0, x_0) + L(\theta - t_0)\Delta \tag{2.10}$$

Since the approximation scheme (2.2), (2.3) generating the function u converges to the value function w and a convergence estimate is given by the quantity $C\Delta^{1/2}$ [8], inequality (2.10) implies the inequality

$$\sigma(x(\theta)) \leq w(t_0, x_0) + L(\theta - t_0)\Delta + C\Delta^{1/2} \tag{2.11}$$

Fixing an arbitrary number $\varepsilon > 0$, one can indicate what stepsize Δ of the partition Γ will guarantee the validity of the estimate

$$\sigma(x(\theta)) \leq w(t_0, x_0) + \varepsilon \tag{2.12}$$

3. APPROXIMATION SCHEMES IN A DIFFERENTIAL GAME WITH DISCOUNT

Let us consider a steady control system over the infinite time interval $[0, +\infty)$

$$\dot{x} = f(x, p, q) = A(x) + B(x)p + C(x)q \tag{3.1}$$

$$x \in R^n, \quad p \in p \subset R^p, \quad q \in Q \subset R^q$$

Let $x(\cdot) = \{x(t): t \in [0, +\infty)\}$ be an arbitrary trajectory of system (3.1). We will estimate the quality of the trajectory by an integral functional with discount coefficient

$$J(x(\cdot), p(\cdot), q(\cdot)) = \int_0^{+\infty} e^{\lambda\tau} g(x(\tau), p(\tau), q(\tau))d\tau, \quad \lambda > 0 \tag{3.2}$$

The functions $f(\cdot)$, $g(\cdot)$ are continuous jointly in all the variables, satisfy Lipschitz conditions with respect to x and are bounded by a constant K.

The stationary value function w^0: $R^n \to R$ in the game (3.1), (3.2) satisfies the equation

$$-\lambda w^0 + H\left(x, \frac{\partial w^0}{\partial x}\right) = 0, \quad x \in R^n \tag{3.3}$$

at its points of differentiability. The function $H(x, s)$: $R^n \times R^n \to R$ in Eq. (3.3) is a Hamiltonian for system (3.1) and is related to the dynamics $f(x, p, q)$ by

$$H(x, s) = \min_{p\in P} \max_{q\in Q} \{\langle s, f(x, p, q)\rangle + g(x, p, q)\} =$$

$$= \langle s, A(x)\rangle + \min_{p\in P} \max_{q\in Q} \{\langle s, B(x)p + C(x)q\rangle + g(x, p, q)\} \tag{3.4}$$

Let us carry over the constructions of the approximation scheme (2.2), (2.3) to the stationary function w^0. The values of a function $u^m_\theta(x)$ approximating the solution $w^0(x)$ are defined by the following iterative procedure

$$u^0_\theta(x) = 0, \quad x \in R^n \tag{3.5}$$

$$u^i_\theta(x) = CUS(u^{i-1}_\theta)(x), \quad i = 0, \dots, m, \quad m = \theta / \Delta \tag{3.6}$$

$$CUS = \min_{y\in \bar{O}(x, K\Delta)} \min_{s\in D^*(G(y))} \{\Delta H(x, e^{-\lambda\Delta}s) + G_\Delta(y) - \langle e^{\lambda\Delta}s, y - x\rangle\}$$

where $G_\Delta(y) = e^{-\lambda\Delta}G(y)$, $D^*G(y)$ is the superdifferential of the local concave closure $G(y)$ for $u^{i-1}_\theta(y)$, $y \in \bar{O}(x, K\Delta)$.

Theorem 3.1. A function $\theta = \theta(x)$ exists such that the approximation $u^m_\theta(\cdot)$ defined by (3.5) and (3.6) converges uniformly to a function w^0: $R^n \to R$—a generalized solution of Eq. (3.3)—as $\Delta \to 0$. Under these conditions, the following estimate holds for some $\tau \in [t_0, +\infty)$

$$\sup||w^0(x) - u_\theta^m(x)|| \leq C\Delta^{\gamma/2} \tag{3.7}$$

The numbers C and $\gamma \in (0, 1)$ are determined by the condition of Hölder continuity [12].

Proof. We will reduce the steady system with integral payoff functional to a system of the form (1.3), (1.4), by adding to system (3.1) an $(n + 1)$th equation which gives the functional (3.2) in differential form

$$\dot{\xi} = k(t, \xi, p, q) = \begin{Vmatrix} \dot{x} \\ \dot{z} \end{Vmatrix} = \begin{Vmatrix} f(x, p, q) \\ e^{-\lambda t} g(x, p, q) \end{Vmatrix} \tag{3.8}$$

$$\xi(t_0) = \begin{Vmatrix} x(t_0) \\ z(t_0) \end{Vmatrix} = \xi_0 = \begin{Vmatrix} x_0 \\ z_0 \end{Vmatrix}$$

$$\xi = (x, z) \in R^{n+1}, \quad t \in [0, +\infty], \quad p \in P, \quad q \in Q, \quad \lambda > 0$$

The payoff functional is defined by

$$J^*(\xi(\cdot)) = \lim_{\theta \to \infty} z(\theta) \tag{3.9}$$

where $z(\theta)$ is the value of the $(n + 1)$th coordinate of the motion $\xi(\cdot)$ of system (3.8) at time θ. Note that when $z_0 = 0$ the functionals J of (3.2) and J^* of (3.9) have the same values.

Let us consider the game (3.8), (3.9) in classes of positional strategies of player I $(t, \xi) \to U(t, \xi)$: $R^{N+1} \to P$ and counter-strategies $(t, \xi, p) \to V(t, \xi, p)$: $R^{n+1} \times P \to Q$ of player II. The sets of strategies U and V will be denoted by **U** and **V**, respectively. Basing ourselves on well-known results [1, 2], we can show that the value of the game (3.8), (3.9) is

$$w^\xi(t_0, \xi_0) = \inf_U \sup_{\xi_1(\cdot)} \lim_{\theta \to +\infty} z_1(\theta) = \sup_C \inf_{\xi_2(\cdot)} \lim_{\theta \to +\infty} z_2(\theta) \tag{3.10}$$

$$U \in \mathbf{U}, \quad V \in \mathbf{V}; \quad \xi_1(\cdot) = \begin{Vmatrix} x_1(\cdot) \\ z_1(\cdot) \end{Vmatrix}; \quad \xi_2(\cdot) = \begin{Vmatrix} x_2(\cdot) \\ z_2(\cdot) \end{Vmatrix}$$

The supremum over $\xi_1(\cdot)$ and infimum over $\xi_2(\cdot)$ are calculated for sets $X_1(t_0, \xi_0, U)$, $X_2(t_0, \xi_0, V)$ whose elements are the motions $\xi_1(\cdot)$, $\xi_2(\cdot)$ of system (3.8) generated by a strategy U or counter-strategy V.

We know [12] that the relation between the value functions $w^0(x)$ and $w^\xi(t, \xi)$ is given by

$$w^\xi(t, \xi) = w^\xi(t, x, z) = z + e^{-\lambda t} w^0(x), \quad w^0(x) = w^\xi(0, x, 0) \tag{3.11}$$

$$x \in R^n, \quad z \in R, \quad t \in [0, +\infty)$$

Consider a time interval $T = [0, \theta]$. Let w_θ^ξ: $T \times R^{n+1} \to R^1$ be the value function in a game of finite duration $\theta(\theta \in [t_0, +\infty])$ with dynamics (3.8) and payoff functional

$$J_\theta^*(\xi(\cdot)) = z(\theta) \tag{3.12}$$

The value functions w^ξ, w_θ^ξ satisfy the estimate

$$\sup||w^\xi(t, \xi) - w_\theta^\xi(t, \xi)|| \leq K\lambda^{-1} e^{-\lambda\theta}, \quad \theta \in [0, +\infty) \tag{3.13}$$

The function w^ξ_θ is the value of the game (3.8), (3.12) with fixed finishing time. We may therefore approximate it using a retrograde procedure with finite-difference operator CU as in (2.1). Divide the time T into m equal parts of length $\Delta > 0$, so that $\theta = m\Delta$. At time θ the approximation $u_\theta^\xi(t, \cdot)$ is defined by the formula

$$u_\theta^\xi(\theta,\xi)=z,\quad (\theta,\xi)\in R^{n+1},\quad z\in R^1 \tag{3.14}$$

Let us assume now that the approximation has already been constructed for time $(i+1)\Delta$, say $u_\theta^\xi((i+1)\Delta,\cdot)$ $(i=0,\dots,m-1)$. Define the approximation $u_\theta^\xi(i\Delta,\cdot)$ at time $i\Delta$ by the formula

$$u_\theta^\xi(i\Delta,\xi)=\max_{\eta\in\bar{O}(\xi,2K\Delta)}\max_{l\in D^*G(\eta)}\{\Delta H(i\Delta,\xi,l)+G(\eta)-\langle l,\eta-\xi\rangle\},\ \xi\in R^{n+1},\ r>2K \tag{3.15}$$

The function $\eta\to G(\eta)$: $\bar{O}(\xi,r\Delta)\to R$ is the local concave closure of the function $u_\theta^\xi(i\Delta,\xi)$. The set $D^*G(\eta)$ is the superdifferential of $G(\eta)$ at the point η. The function $H(t,\xi,l)=R^{n+1}\times R^{n+1}\to R$ in Eq. (3.15) is the Hamiltonian of the extended system (3.8) and is related to the dynamics $k(t,\xi,p,q)$ by

$$H(t,\xi,l)=\min_{p\in P}\max_{q\in Q}\langle l,k(t,\xi,p,q)\rangle \tag{3.16}$$

By (3.11), the value function, hence also its local convex closure, are linear with respect to the $(n+1)$th variable z. Therefore the vector l has the structure $l=(s,l)$ and the Hamiltonian is defined by

$$H(t,\xi,l)=\min_{p\in P}\max_{q\in Q}\{\langle s,f(x,p,q)\rangle+e^{-\lambda t}g(x,p,q)\}=$$
$$=\langle s,A(x)\rangle+\min_{p\in P}\max_{q\in Q}\{\langle s,B(x)p+C(x)q\rangle+e^{-\lambda t}g(x,p,q)\}$$

Comparing the formulae for the approximations $u_\theta^\xi(t,\cdot)$ of (3.14), (3.15) and u_θ^m of (3.5), (3.6), we obtain

$$u_\theta^\xi(t,\xi)=z+e^{-\lambda t}u_\theta^m(x) \tag{3.17}$$

It is well known [7, 8] that an approximation scheme over a finite time interval T ensures convergence, with the estimate

$$\sup\|w_\theta^\xi(0,x,0)-u_\theta^m(x)\|\leq\Delta^{1/2}L\int_0^\theta e^{(L-\lambda)\tau}d\tau \tag{3.18}$$

Taking (3.11), (3.17) and (3.13), (3.18) into consideration, we obtain the inequality

$$\sup\|w^0(x)-u_\theta^m(0,x)\|\leq\Delta^{1/2}L\int_0^\theta e^{(L-\lambda)\tau}d\tau+K\lambda^{-1}e^{-\lambda\theta} \tag{3.19}$$

Minimizing the right-hand side of (3.19) with respect to θ, we obtain the desired relationship (3.7).

4. SYNTHESIS OF STEADY STRATEGIES

Let us assume that the function $u^m{}_\theta$ has been constructed using the approximation scheme (3.5), (3.6). We define the value of the positional control $U^*=U^*(x)$ at a point $x\in R^n$ as follows:

$$U^*=U^*(x)=\arg\min_{p\in P}\max_{q\in Q}\{\langle s^*,f(x,p,q)\rangle+g(x,p,q)\}=$$

$$=\arg\min_{p\in P}\{\langle s^*,B(x)p\rangle+\max_{q\in Q}\langle s^*,C(x)q\rangle+g(x,p,q)\}\} \tag{4.1}$$

$$s^*=s^*(x,y^*)=\arg\min_{s\in D^*(G(y))}\{\Delta H(x,e^{-\lambda\Delta}s)+G_\Delta(y^*)-\langle e^{-\lambda\Delta}s,y^*-x\rangle\} \tag{4.2}$$

$$y^*=y^*(x)=\arg\min_{y\in\bar{O}(x,K\Delta)}\min_{s\in D^*(G(y))}\{\Delta H(x,e^{-\lambda\Delta}s)+G_\Delta(y)-\langle e^{-\lambda\Delta}s,y-x\rangle\} \tag{4.3}$$

For points

$$\xi=(x,z),\quad \eta(t,\xi,\Delta,U^*,q)=(y(x,\Delta,U^*,q),\ z(t,x,\Delta,U^*,q))$$

$$y(x,\Delta,U^*,q)=x+\Delta(A(x)+B(x)U^*+C(x)q)$$

$$z(t,x,\Delta,U^*,q))=z+\Delta e^{-\lambda t}g(x,U^*,q),\quad q\in Q$$

we will analyse the relation between the values $u^m_\theta(x)$ and $u^m_\theta(y(x,\Delta,U^*,q))$ of the approximation function u^m_θ.

Lemma 4.1. The control U^* of (4.1) satisfies the inequality

$$\max_{q\in Q}\{z(t,x,\Delta,U^*,q)+e^{-\lambda(t+\Delta)}u^m_\theta(y(t,x,\Delta,U^*,q))\}\leq z+e^{-\lambda t}u^m_\theta(x) \tag{4.4}$$

Proof. Consider the function G^* conjugate to the local concave closure $\eta\to G(\eta)$: $\overline{O}(\xi,r\Delta)\to R$. Since G and G^* are concave functions, we have the following chain of inequalities for any $q\in Q$

$$u^\xi_\theta(t+\Delta,\eta(t,\xi,\Delta,U^*,q))\leq G(\eta(t,\xi,\Delta,U^*,q))\leq \max_{q\in Q}G(\eta(t,\xi,\Delta,U^*,q))=$$

$$=\max_{q\in Q}\inf_{l\in R^{n+1}}\{\langle l,\eta\rangle-G^*\}=\max_{q\in Q}\min_{\eta\in\overline{O}(\xi,2K\Delta)}\min_{l\in D^*G(\eta)}\{\langle l,\eta\rangle-\inf_{\eta\in R^{n+1}}\{\langle l,\eta\rangle-G(\eta)\}\} \tag{4.5}$$

It follows from property (3.11) of the value function of the extended system (3.8) that its local concave closure G satisfies the relation

$$G(\eta(t,\ \xi,\ \Delta,\ U^*,\ q))=z(t,\ x,\ \Delta,\ U^*,\ q)+e^{-\lambda(t+\Delta)}G(y(t,\ x,\ \Delta,\ U^*,\ q))$$

and the vector l has the structure

$$l=(e^{-\lambda(t+\Delta)}s,\ 1)$$

Hence

$$\left\langle l,\ \eta\right\rangle=\left\langle e^{-\lambda(t+\Delta)}s,\ x\right\rangle+\Delta\left(\left\langle e^{-\lambda(t+\Delta)}s,\ A(x)\right\rangle+\right.$$

$$\left.+\left\langle e^{-\lambda(t+\Delta)}s,\ B(x)U^*\right\rangle+\left\langle e^{-\lambda(t+\Delta)}s,\ C(x)q\right\rangle\right)+z+\Delta e^{-\lambda t}g(x,\ U^*,\ q)) \tag{4.6}$$

Note that for $l\in D^*G(\eta)$ the conjugate function $G^*(\eta)$ satisfies the equality

$$\inf_{\eta\in R^{n+1}}\left\{\left\langle l,\ \eta\right\rangle-G^*(\eta)\right\}=\left\langle e^{-\lambda(t+\Delta)}s,\ y\right\rangle-e^{-\lambda(t+\Delta)}G(y) \tag{4.7}$$

Using the Minimax theorem and Eqs (4.5)–(4.7), we obtain the inequality

$$u^\xi_\theta(t+\Delta,\ \eta(t,\ \xi,\ \Delta,\ U^*,\ q))\leq z+e^{-\lambda t}\min_{y\in O(x,K\Delta)}\min_{s\in D^*(G(y))}\left(\left\langle e^{-\lambda\Delta}s,\ x-y\right\rangle+\right.$$

$$+G_\Delta(y)+\Delta\left(\left\langle e^{-\lambda\Delta}s,\ A(x)\right\rangle+\left\langle e^{-\lambda\Delta}s,\ B(x)U^*\right\rangle+\right.$$

$$\left.\left.+\max_{q\in Q}\left\{\left\langle e^{-\lambda\Delta}s,\ C(x)q\right\rangle+g(x,\ U^*,\ q)\right\}\right)\right) \tag{4.8}$$

Taking the definition of the positional control U^* (4.1)–(4.3) into consideration, we obtain inequality (4.4).

Let us estimate the quality of the entire trajectory $x(\cdot)$ generated by the strategy U^* of (4.1) and an arbitrary perturbation $\tau\to q(\tau)$

$$x(\cdot)=\{x(t,\ x_0,\ U^*,\ q(\cdot)),\ t\in\Gamma\bigcap[0,\ +\infty]\}$$
$$x(t_{i+1})=x(t_i+\Delta)=x(t_i)+\Delta(A(x(t_i))+B(x(t_i))U^*+C(x(t_i))q(t_i)),\tag{4.9}$$
$$t_i,\ t_{i+1}\in\Gamma,\quad x(0)=x_0$$

Theorem 4.1. For any partitions Γ, initial positions x_0 and arbitrary perturbations $\tau\to q(\tau)$, the trajectory $x(\cdot)$ of (4.9) generated by the strategy U^* defined by (4.1) satisfies the estimate

$$J(x(\cdot),\ U^*,\ q(\cdot))<w^0(x_0)+C\Delta^{\gamma/2}\tag{4.10}$$

Fixing an arbitrary number $\varepsilon>0$, we can find a step size Δ for the partition Γ such that

$$J(x(\cdot),\ U^*,\ q(\cdot))<w^0(x_0)+\varepsilon\tag{4.11}$$

The proof of the theorem is analogous to the proofs of Theorems 2.1 and 3.1.

5. THE RELATION BETWEEN STEP SIZES OF THE FINITE-DIFFERENCE APPROXIMATION SCHEME

In reality, the approximation procedure (2.2), (2.3) cannot be implemented at every point $(t,x)\in G_r$, $t\in\Gamma$, but only at mesh points of a grid $GR(t)$. Let us assume that the grid $GR(t)$, $t\in\Gamma$, is uniform and rectangular

$$GR(t)=\{x\in R^n:(t,\ x)\in G_r,\ x=\sum(m_1e_1+\ldots+m_ne_n)\gamma\Delta\}\tag{5.1}$$
$$m_i=0,\ \pm1,\ \pm2,\ \ldots,\ i=1,\ \ldots,\ n$$
$$e_i=(e_i^1,\ \ldots,\ e_i^n),\ e_i^i=1,\ e_i^j=0,\ i=1,\ \ldots,\ n,\ i\neq j$$

The values of the operator CU will be defined only at mesh points y_j of $GR(t)$ and will then be determined by linear interpolation relative to a given simplicial partition Ω

$$CU(t,\ \Delta,\ w)(y)=\sum_{j=0}^{n}\alpha_iCU(t,\ \Delta,w)(y_j)$$
$$y=\sum\alpha_jy_j,\ y_j\in GR(t),\ \alpha_j\geqslant0,\ j=0,\ \ldots,\ n,\ \sum\alpha_j=1\tag{5.2}$$

The numbers $\alpha_j=\alpha_j(\Omega)$ and mesh points $y_i=y_i(\Omega)$ depend on the partition Ω.

Let us consider an approximation scheme with finite-difference operator CU of (5.2) for a partition Γ of the interval T with step size Δ

$$u^*(\theta,\ y)=\sigma(y)=\sum\alpha_j\sigma(y_j),\ y=\sum\alpha_jy_j$$
$$\sum\alpha_j=1,\ \alpha_j=\alpha_j(\Omega)\geqslant0,\ y_j=y_j(\Omega)\in GR(\theta),\ j=0,\ \ldots,\ n$$
$$u(t,\ x)=CU(t,\ t_{i+1}-t,\ u(t_{i+1},\ \cdot))(x)$$
$$t\in[t_i,\ t_{i+1}),\ i=0,\ \ldots,\ N-1$$

We first define the control $U^*=U^*(t,x)$ at mesh points $x\in GR(t)$, $t\in\Gamma$, using the operator $CU(t,\Delta,u(t+\Delta),\cdot))(x)$ of (2.4)–(2.6). A strategy $U^c(t,y)$ is defined at the point y by piecewise-constant interpolation of the values $\{U^*(t,x),x\in GR(t),t\in\Gamma\}$ of (2.4), computed at the nearest mesh points x of the grid $GR(t)$

$$U^c(t,\ y)=U^*(t,\ x),\quad x=x(y)=\arg\min_{z\in GR(t)}\|y-z\|\tag{5.3}$$

Consider the Euler polygon

$$y(\cdot)=\{y(t,\ t_0,\ y_0,\ U^c,\ q),\ \ t\in\Gamma\} \tag{5.4}$$

generated by the strategy U^c of (5.3) and an arbitrary perturbation $\tau\to q(\tau)$

$$y(t_{i+1})=y(t_i+\Delta)=y(t_i)+\Delta(A(t_i,\ y(t_i))+B(t_i,\ y(t_i))U^c+$$
$$+C(t_i,\ y(t_i))q(t_i)),\ \ t_i,\ t_{i+1}\in\Gamma,\ \ y(t_0)=y_0$$

For the trajectory $y(\cdot)$ of (5.4), we define associated points $x_-(t_i)$, $x_+(t_{i+1})$ by the relations

$$x_-(t_i)=x(y(t_i))=\arg\min_{z\in GR(t)}\|y(t_i)-z\| \tag{5.5}$$

$$x_+(t_{i+1})=x_-(t_i)+\Delta(A(t_i,\ x_-(t_i))+B(t_i,\ x_-(t_i))U^c+C(t_i,\ x_-(t_i))q(t_i)),\ \ t_i,\ t_{i+1}\in\Gamma \tag{5.6}$$

Lemma 5.1. The trajectory $y(\cdot)$ of (5.4) and associated points $x_-(t_i)$, $x_+(t_{i+1})$ of (5.5), (5.6) satisfy the estimate

$$\|y(t_{i+1})-x_+(t_{i+1})\|<(1+L\Delta)\|y(t_i)-x_-(t_i)\| \tag{5.7}$$

Lemma 5.2. Let the parameter γ of the grid $GR(t)$, $t\in\Gamma$, be infinitesimal relative to the partition step size Δ

$$\gamma=\varepsilon(\Delta),\ \ \lim_{\Delta\to 0}\varepsilon(\Delta)=0$$

for example

$$\gamma=\rho\Delta^a,\ \ a>0,\ \rho>0 \tag{5.8}$$

that is, the step size h of $GR(t)$, which is a grid on the phase variables x (i = 1, . . . , n), is an infinitesimal

$$h=\rho\Delta^{1+a} \tag{5.9}$$

small to a higher order than the step size Δ of the partition Γ of the time interval T.

Then

$$u(t_i,\ y(t_i))\geqslant u(t_{i+1}),\ y(t_{i+1}))-L_w\frac{n^{1/2}}{2}(2+L\Delta)\rho\Delta^a\Delta \tag{5.10}$$

Proof. It follows from the Lipschitz continuity of u and relations (2.7) and (5.7) that

$$u(t_i,\ y(t_i))>u(t_i,\ x_-(t_i))-L_w\|y(t_i)-x_-(t_i)\|>u(t_{i+1},\ x_+(t_{i+1}))-L_w\|y(t_i)-x_-(t_i)\|>$$
$$>u(t_{i+1},y(t_{i+1}))-L_w(2+L\Delta)\|y(t_i)-x_-(t_i)\|$$

It follows from (5.8) that

$$\|y(t_i)-x_-(t_i)\|<\frac{n^{1/2}}{2}\rho\Delta^a\Delta \tag{5.11}$$

The last two inequalities imply (5.10).

Using (5.10), one can prove the following proposition for a trajectory $y(\cdot)$ generated by a positional control (5.3).

Theorem 5.1. For any partitions Γ, grids $GR(t)$, $t\in\Gamma$ with parameters (5.8) of a high order of smallness, initial positions (t_0, y_0) and arbitrary perturbations $\tau\to q(\tau)$, the trajectory $y(\cdot)$ of (5.4) generated by a strategy U^c with piecewise-constant interpolation satisfies the estimate

$$\sigma(y(\theta))\leqslant u(t_0,y_o)+\varphi(\Delta) \tag{5.12}$$

$$\varphi(\Delta)=(\theta-t_*)L_w\frac{n^{1/2}}{2}(2+L\Delta)\rho\Delta^a,\quad \lim_{\Delta\to 0}\varphi(\Delta)=0$$

and consequently

$$\sigma(y(\theta))<w(t_0,\ y_0)+C^*\Delta^{1/2}+\varphi(\Delta) \tag{5.13}$$

Fixing an arbitrary number $\varepsilon > 0$, one can find a step size of the partition Γ such that

$$\sigma(y(\theta))\leqslant w(t_0,\ y_0)+\varepsilon \tag{5.14}$$

Remark 5.1. There are other possible piecewise-constant interpolations of the strategy $U^* = U^*(t,x)$ of (2.4). For example, the strategy $U^e(t,y)$ may be completed in accordance with a simplicial partition Ω by the values of the controls $U^*(t,x)$ computed at the mesh points x of $GR(t)$ with least values of the function u

$$U^e(t,\ y)=U^*(t,\ x),\quad x=x(y)=\arg\min_{y_j}u(y_j) \tag{5.15}$$

$$y_j\in GR(t),\ \ \Sigma\alpha_j y_j=y,\ \ \Sigma\alpha_j=1,\ \alpha_j\geqslant 0,\ \alpha_j=\alpha_j(\Omega)$$

In that case, estimates (5.10) and (5.13) must be rewritten in the form

$$u(t_i,\ y(t_i))\geqslant u(t_{i+1},\ y(t_{i+1}))-L_w n^{1/2}(1+L\Delta)\rho\Delta^a\Delta \tag{5.16}$$

$$\sigma(y(\theta))<w(t_0,\ y_0)+C^*\Delta^{1/2}+\varphi(\Delta) \tag{5.17}$$

$$\varphi(\Delta)=(\theta-t_0)L_w n^{1/2}(1+L\Delta)\rho\Delta^a$$

Note that analogous results may be obtained from the trajectory $y(\cdot)$ in a steady problem with discount (3.1).

Theorem 5.2. For any partitions Γ, grids $GR(t)$, $t\in\Gamma$, with parameters (5.8) of a high order of smallness, initial positions y_0 and arbitrary perturbations $\tau\to q(\tau)$, the trajectory $y(\cdot)$ generated by a strategy U^e of (5.3) with piecewise-constant interpolation satisfies the estimate

$$J(y(\cdot),\ U^c,\ q(\cdot))<w^0(y_0)+C(\Delta^{1/2}+R(\Delta)\Delta^a)^{\lambda/(L+1)} \tag{5.18}$$

$$R(\Delta)=\frac{n^{1/2}}{2}\rho(2+3L\Delta)$$

Fixing an arbitrary number $\varepsilon > 0$, one can find a step size Δ for Γ such that

$$J(y(\cdot),\ U^c,\ q(\cdot))<w^0(y_0)+\varepsilon \tag{5.19}$$

Proof. We again turn to the approximation function u^ξ_θ of the extended system (3.8). Under the assumptions of the lemma, the following inequality holds for u^ξ_θ and the trajectory $\eta(\cdot) = (y(\cdot), z(\cdot))$

$$u^\xi_\theta(t_i,\ \eta(t_i))>u^\xi_\theta(t_{i+1},\ \eta(t_{i+1}))-L^\xi_w\frac{n^{1/2}}{2}(2+3L\Delta)\rho\Delta^{a+1}$$

where the Lipschitz constant for the approximation function is defined by

$$L^\xi_w=\frac{L}{L-\lambda}(e^{(L-\lambda)\theta}-1)$$

Then the following inequality holds on the whole trajectory $\eta(\cdot)$

$$u^\xi_\theta(t_0,\ \eta(t_0))>J^*_\theta(\cdot)-R(\Delta)\Delta^a\theta(e^{(L-\lambda)\theta}-1)$$

Using the same technique as in the proof of Theorem 2.1, we obtain an estimate

$$J(y(\cdot)) < w^0(y_0) + \phi(\theta, \Delta)$$

$$\phi(\theta, \Delta) = \frac{2K}{\lambda}e^{-\lambda\theta} + \Delta^{1/2}\frac{L}{L-\lambda}(e^{(L-\lambda)\theta} - 1) + R(\Delta)\Delta^a\frac{L}{L-\lambda}(e^{(L-\lambda)\theta} - 1)$$

If $L - \lambda > 0$, the function $\phi(\theta, \Delta)$ becomes infinitely small as $\Delta \to 0$, $\theta \to \infty$. Otherwise, we replace $\phi(\theta, \Delta)$ by a larger function

$$\phi(\theta, \Delta) = \frac{2K}{\lambda}e^{-\lambda\theta} + (\Delta^{1/2} + R(\Delta)\Delta^a)\frac{L}{L-\lambda}(e^{(L-\lambda)\theta} - 1)$$

The minimum point of $\phi(\theta)$ as a function of θ is

$$e^\theta = \left(\frac{2K(L-\lambda)}{L(L-\lambda+1)(\Delta^{1/2} + R(\Delta)\Delta^a)}\right)^{1/(L+1)} \tag{5.20}$$

This relation determines the form of the second term in Theorem 5.2.

6. COMPUTATIONAL EXPERIMENTS

In biological processes [13], the evolution of the interaction of two populations develops in accordance with the accumulation of experience and repeated situations, where each participant acts in keeping with one of two behaviour modes that are possible for its population. Let x ($0 \leqslant x \leqslant 1$) be the frequency of those individuals in the first population which, at a given instant of time, have adopted the first behaviour mode (strategy), so that the frequency of those adopting the second mode is $1 - x$. The parameters y and $1 - y$ have a similar interpretation for the second population. Let us assume that the interests of the populations are given by a payoff matrix

$$A = \begin{Vmatrix} 11 & 2 \\ 3 & 6 \end{Vmatrix}, \quad B = \begin{Vmatrix} 2 & 4 \\ 5 & 1 \end{Vmatrix}$$

The average payoffs of the coalition are defined by the functions

$$g_1 = C_a xy - \alpha_1 x - \alpha_2 y + a_{22}, \quad g_2 = C_b xy - \beta_1 x - \beta_2 y + b_{22}$$
$$C_a = 12, \quad \alpha_1 = 4, \quad \alpha_2 = 3, \quad C_b = -6, \quad \beta_1 = -3, \quad \beta_2 = -4$$

The process is simulated by a replicator dynamics [13]

$$\dot{x} = x(1-x)(C_a y - \alpha_1), \quad \dot{y} = y(1-y)(C_b x - \beta_2) \tag{6.1}$$

in which the rates of change of the groups employing the different strategies in one population are linearly related to the population payoffs

$$\ln\left(\frac{x}{1-x}\right)' = (C_a y - \alpha_1) = \frac{\partial g_1}{\partial x}, \quad \ln\left(\frac{y}{1-y}\right)' = (C_b x - \beta_2) = \frac{\partial g_2}{\partial y} \tag{6.2}$$

An analysis of an arbitrary evolutionary system with smooth dynamics, including the replicator dynamics (6.1), may be found in [14]. Here we propose to investigate a dynamical game formulation in which the controlling strategies may be discontinuous functions of the position.

Let us assume that the rates of change p and q of the population structure are the control parameters

$$\dot{x} = x(1-x)p, \quad \dot{y} = y(1-y)q \tag{6.3}$$

and that p and q obey constraints related to the replicator dynamics (6.1)

$$p \in P, \quad P = [\min\{0, C_a\} - \alpha_1, \ \max\{0, C_a\} - \alpha_1]$$

$$q \in Q, \quad Q = [\min\{0, C_b\} - \beta_2, \ \max\{0, C_b\} - \beta_2]$$

Consider the problem of constructing optimal guaranteeing strategies in a non-autonomous game, as well as the Nash equilibrium trajectory generated by those strategies.

We define the payoff of each coalition by an integral functional with discount coefficient λ. This functional may be considered as a global payoff over the infinite time interval $[0, +\infty]$

$$J_i = \int_0^{+\infty} e^{-\lambda t} g_i(x(t),\ y(t))dt, \quad i = 1,\ 2 \tag{6.4}$$

To construct optimal controls of the game just formulated, we must consider two guaranteed problems of the same type, for the functionals J_1 and J_2 [15]. Let us consider, say, the first such problem. The value function $(x, y) \to w_1(x, y)$ is a solution of the Hamilton–Jacobi equation

$$-\lambda w_1(x,\ y) + g_1(x,\ y) + x(1-x)\max_{p\in P}\frac{\partial w_1}{\partial x}p + y(1-y)\min_{q\in Q}\frac{\partial w_1}{\partial y}q = 0 \tag{6.5}$$

An approximation scheme to solve this equation is defined as follows.

Consider a time interval T and a partition $\Gamma = [t_0 = 0 < t_1 < \ldots < t_m = \theta]$ of stepsize Δ. We define the approximation function W by an iterative procedure, as follows. Set $W(T, x, y) = 0$. Suppose that at some $t + \Delta$ we have already defined $W(t + \Delta, x, y)$. At time t we define $W(t, x, y)$ by

$$W(t,\ x,\ y) = \max_{u\in P}\ \min_{v\in Q}\{\Delta g_1(x,\ y) + (1-\lambda\Delta)W(t+\Delta,\ x+\Delta x(1-x)p,\ y+\Delta y(1-y)q)\}$$

At $t = 0$ we obtain an approximation $W(0, x, y)$ for the solution $w_1(x, y)$.

In parallel with the computation of the value function, we construct a maximizing strategy u^0 for the first coalition. The structure of the strategy is shown in Fig. 1. The square of the phase state is divided into three regions. In the first region, above the curve L_1, the strategy u^0 takes its extreme value $u^0 = C_a - \alpha_1 = 8$. This means that for maximum success of the population, members of the population must be directed at a maximum rate to the first behaviour mode. In the third region, beneath the curve L_2, the value of u^0 takes the other extreme value $u^0 = -\alpha_1 = -4$, and members are directed to the second mode. The region numbered 2, between the curves L_1 and L_2, represents an intermediate layer, in which the values of the control do not attain their extreme values.

An approximation of the value function $w_2(x, y)$ and a maximizing strategy of the second coalition υ^0 in the game with payoff matrix B are constructed in the same way.

Figure 2 shows the intermediate layers S_u for the strategy u^0, $S\upsilon$ for the strategy υ^0, and an equilibrium trajectory (the curve with arrows) generated by strategies u^0, υ^0 for the initial position IP = (0.1, 0.95). This trajectory is the main construction of the dynamical Nash equilibrium proposed in [15]. The results of these computational experiments show that the equilibrium trajectories for different initial positions converge to the stationary point DE = (0.729, 0.496) at which the rates of the controlled system (6.3) vanish: $\dot{x} = 0, \dot{y} = 0$.

Note that the trajectories of classical models with replicator dynamics (6.1) converge to the static Nash equilibrium point NE = (0.67, 0.33) or circulate in its neighbourhood. The value of the payoff functions $g_i(x, y)$ $(i = 1, 2)$ at the point DE is better (strictly greater) than at the static equilibrium point NE. Consequently, the values of the payoff functionals J_i $(i = 1, 2)$ on trajectories converging to a dynamical equilibrium point are better than on trajectories converging to a static equilibrium point.

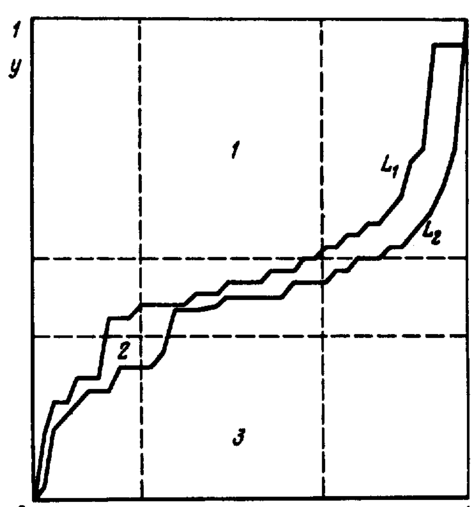

Fig. 1.

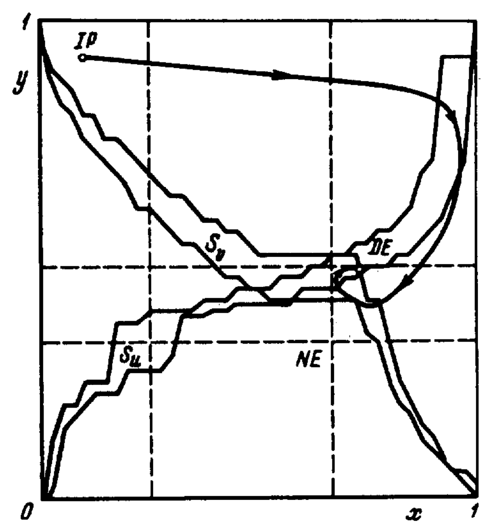

Fig. 2.

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